William Lawvere & The Paradox Generating Machine

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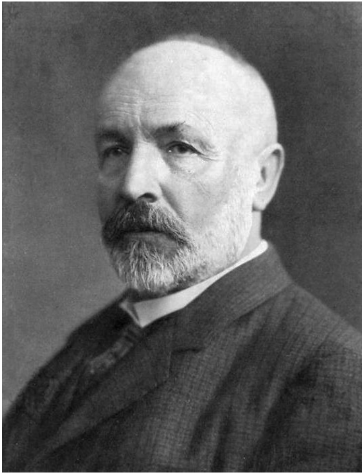
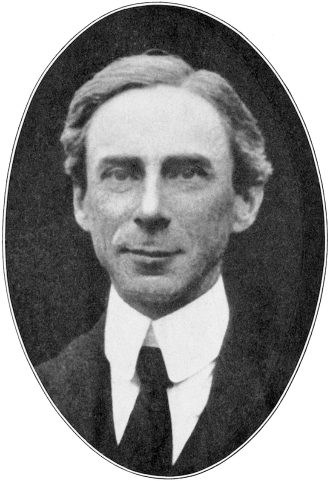
24 April 2019



Georg Cantor



Bertrand Russell

Cantor's Theorem

There is no surjective function $f : \mathbb{N} \to \mathbf{2}^{\mathbb{N}}$.

Proof.

Such a function f would define an enumeration $S_1, S_2, \ldots, S_n, \ldots$ of subsets of $\mathbb{N}$:

$$
\begin{array}{rcllllll}
S_1 & = & \{\mathbf{1}, & 2, & 3, & 4, & 5, & \ldots\} \\
S_2 & = & \{1, & , & 3, & 4, & 5, & \ldots\} \\
S_3 & = & \{1, & 2, & \mathbf{3}, & 4, & 5, & \ldots\} \\
S_4 & = & \{1, & 2, & 3, & , & 5, & \ldots\} \\
S_5 & = & \{1, & 2, & 3, & 4, & , & \ldots\} \\
\ldots & & & & & & & \\
G & = & \{\ , & \mathbf{2}, & , & \mathbf{4}, & \mathbf{5}, & \ldots\}
\end{array}
$$

However, G cannot occur in the enumeration, since it differs from each S_n at the nth place. Hence, there is no such function f. □

Cantor's Theorem - The Statement

There is no surjective function $f : \mathbb{N} \to \mathbf{2}^{\mathbb{N}}$.

- $\mathbf{2} = \{0, 1\}$, $\mathbf{n} = \{0, 1, \ldots, (n-1)\}$.

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$$S \subseteq \mathbb{N} \;\leftrightarrow\; g : \mathbb{N} \to \mathbf{2}$$

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Cantor's Theorem - The Statement

There is no "surjective" function $f : \mathbb{N} \times \mathbb{N} \to \mathbf{2}$.

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Surjectivity

Definition

A function $f\colon X \to Y$ is surjective iff

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

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A function $\phi\colon X \times X \to Y$ is "surjective" iff

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A function $\phi\colon X \times X \to Y$ is "surjective" iff

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A function $\phi \colon X \times X \to Y$ is *not representable* by some $g \colon X \to Y$ iff

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A function $\phi\colon X \times X \to Y$ is *not representable* by some $g\colon X \to Y$ iff

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The Diagonal Argument — Power Set Version

Suppose that there is such a function f.
Then f defines an enumeration $S_1, S_2, \ldots, S_n, \ldots$ of subsets of $\mathbb{N}$:

$$
\begin{array}{rcllllll}
S_1 & = & \{\mathbf{1}, & 2, & 3, & 4, & 5, & \ldots\} \\
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G & = & \{\;\;, & \mathbf{2}, & \;\;, & \mathbf{4}, & \mathbf{5}, & \ldots\}
\end{array}
$$

However, G cannot occur in the enumeration, since it differs from each S_n at the nth place. Hence, there is no such function f.

The Diagonal Argument — Uncountability Version

Such a function f also defines an enumeration of infinite sequences of binary digits:

$$
\begin{array}{rl}
S_1 = & (\mathbf{1} \quad 1 \quad 1 \quad 1 \quad 1 \quad \dots) \\
S_2 = & (1 \quad \mathbf{0} \quad 1 \quad 1 \quad 1 \quad \dots) \\
S_3 = & (1 \quad 1 \quad \mathbf{1} \quad 1 \quad 1 \quad \dots) \\
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\dots & \\
G = & (\mathbf{0} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \quad \dots)
\end{array}
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The Diagonal Argument

f	1	2	3	4	5	$n \rightarrow$ …
1	1	1	1	1	1	…
2	1	0	1	1	1	…
3	1	1	1	1	1	…
4	1	1	1	0	1	…
5	1	1	1	1	0	…
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\ddots$
m						
$\downarrow$						

$$f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbf{2}, \quad f(n, m) = \begin{cases} 1 & n \in S_m \\ 0 & n \notin S_m \end{cases}$$

The Diagonal Argument — Constructing G

How did we construct G?

- "f defines an enumeration"

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- $f : \mathbb{N} \times \mathbb{N} \to \mathbf{2}$,

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- G "differs from"
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$$f(n, m) = \begin{cases} 1 & n \in S_m \\ 0 & n \notin S_m \end{cases}$$

- $t : \mathbf{2} \to \mathbf{2}$,

$$t(n) = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \end{cases}$$

  Note that t has **no fixed points**.

- "each S_n at the nth place"

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The Diagonal Argument — Constructing G

How did we construct G?

- $f\colon \mathbb{N} \times \mathbb{N} \to \mathbf{2}$,

$$f(n, m) = \begin{cases} 1 & n \in S_m \\ 0 & n \notin S_m \end{cases}$$

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We can compose these to get a function $g\colon \mathbb{N} \to \mathbf{2}$:

$$\mathbb{N} \xrightarrow{\Delta} \mathbb{N} \times \mathbb{N} \xrightarrow{f} \mathbf{2} \xrightarrow{t} \mathbf{2}$$

The Diagonal Argument — The Diagram

We get the following diagram, which commutes:

$$\begin{array}{ccc} \mathbb{N} \times \mathbb{N} & \xrightarrow{f} & \mathbf{2} \\ {\scriptstyle \Delta}\big\uparrow & & \big\downarrow{\scriptstyle t} \\ \mathbb{N} & \xrightarrow{g} & \mathbf{2} \end{array}$$

Note that $g = \chi(G)$ is the characteristic function of the set

$$G = \{n \in \mathbb{N} | n \notin S_n\},$$

$f : \mathbb{N} \times \mathbb{N} \to \mathbf{2}$ is

$$f(n, m) = \begin{cases} 1 & n \in S_m \\ 0 & n \notin S_m \end{cases},$$

and t has **no fixed points**.

The Diagonal Argument

$$\begin{array}{ccc} \mathbb{N} \times \mathbb{N} & \xrightarrow{f} & \mathbf{2} \\ \Delta \uparrow & & \downarrow t \\ \mathbb{N} & \xrightarrow{g} & \mathbf{2} \end{array}$$

Note that g is the characteristic function of the set

$$G = \{n \in \mathbb{N} | n \notin S_n\}.$$

Observe that $\forall n \in \mathbb{N}$, $f(-, n) \neq g(-)$.

Otherwise, if $f(-, n_0) = g(-)$ for some $n_0 \in \mathbb{N}$, then

$$f(n_0, n_0) = g(n_0) = t(f(n_0, n_0)).$$

This shows that t has a fixed point, which is a contradiction. $\square$

Russell's Paradox

Substitute *Sets* for $\mathbb{N}$ and define $f\colon Sets \times Sets \to \mathbf{2}$ to be

$$f(S, T) = \begin{cases} 1 & S \in T \\ 0 & S \notin T \end{cases}.$$

Russell's paradox arises by noting that we cannot construct a similar function $g\colon Sets \to \mathbf{2}$, as that would lead us to conclude that, for some $S \in Sets$,

$$f(S, S) = g(S) = t(f(S, S)),$$

i.e. that $S \in S$ and $S \notin S$ at the same time.

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$$\begin{array}{ccc} Adj \times Adj & \xrightarrow{f} & \mathbf{2} \\ \Delta \uparrow & & \downarrow t \\ Adj & \xrightarrow{g} & \mathbf{2} \end{array}$$

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  - Then $Y = f(A,A) = A(A) = (t(x(x)))(t(x(x)))$.
  - In lambda calculus, $Y = \lambda t.(\lambda x.t(xx))(\lambda x.t(xx))$.

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- William Lawvere, "Diagonal Arguments and Cartesian Closed Categories", Lecture Notes in Mathematics vol. 92 (1969), pp. 134–145.
- William Lawvere, "Diagonal Arguments and Cartesian Closed Categories", *Reprints in Theory and Applications of Categories*, **15** (2006), pp. 1–13.
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