

The roadmap

The aim is to understand this:

In category theory, the concept of catamorphism denotes the unique homomorphism from an initial algebra into some other algebra.

To better understand this:

In functional programming, catamorphisms provide generalizations of folds of lists to arbitrary algebraic data types, which can be described as initial algebras. The dual concept is that of anamorphism that generalize unfolds. A hylomorphism is the composition of an anamorphism followed by a catamorphism.

The outline

- ▶ Basic category theory

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- ▶ Haskell as a category (if you squint)

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- ▶ Initial objects

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- ▶ Initial objects
- ▶ Catamorphisms as unique homomorphisms from initial objects

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- ▶ Category theoretic functors, Haskell Functors endofunctors
- ▶ Algebras over an endofunctor
- ▶ F-Algebra homomorphisms, or arrows in the category of F-algebras
- ▶ Initial objects
- ▶ Catamorphisms as unique homomorphisms from initial objects
- ▶ Flip the arrows

What's a category?

Pick some things:

- ▶ Objects (X, Y, Z)

Assert some properties:

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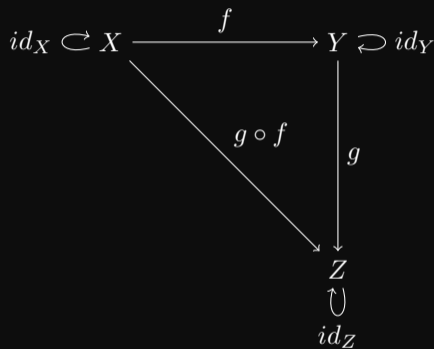


Figure 1:

Haskell as sort of a category

Pick some things:

- ▶ Objects are types (not values!)

Assert some properties:

```
(.) :: (b -> c) -> (a -> b) -> a -> c  
id  :: a -> a
```

NB: it's lies, all lies!

```
seq undefined () = undefined  
seq (undefined . id) () = ()
```


Haskell as sort of a category

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Functors, endofunctors

A functor is a mapping between categories that sends objects to objects (types to types) and arrows to arrows (terms to terms), preserving identity arrows and composition, possibly across categories.

```
fmap id = id
fmap f . fmap g = fmap (f . g)
```

Endofunctors map from a category to the same category.

In the case of Hask:

```
class Functor (f :: * -> *) where
  fmap :: (a -> b) -> f a -> f b
```

Algebra “over” an endofunctor

For a category C and endofunctor F an algebra of F is an object X in C and a morphism:

$$\text{alg} : F(X) \rightarrow X$$

X is called the “carrier” of the algebra.

```
-- For a category and endofunctor
data F a = Zero | Succ a

instance Functor F where
  fmap _ Zero      = Zero
  fmap f (Succ a) = Succ (f a)

-- An algebra of F is an X in C
type X = Natural

-- And a morphism
alg :: F X -> X
alg Zero = 0
alg (Succ n) = n + 1
```

F Natural -> Natural

```
data F a = Zero | Succ a
```

```
alg :: F Natural -> Natural
```

```
alg Zero = 0
```

```
alg (Succ n) = n + 1
```

```
> alg Zero
```

```
0
```

```
> alg $ Succ $ alg Zero
```

```
1
```

```
> alg $ Succ $ alg $ Succ $ alg Zero
```

```
2
```

An alternate algebra, same F and C, different X

```
alg' :: F String -> String
alg' Zero      = "!"
alg' (Succ s) = "QUACK" ++ s
```

```
> alg' $ Succ $ alg' $ Succ $ alg' Zero
"QUACKQUACK!"
```

Prescient fun fact

An initial object of a category C is an object I in C such that for every object X in C , there exists precisely one morphism $I \rightarrow X$. - Wikipedia

(up to isomorphism)

Initial, in the category of algebras

An initial algebra for an endofunctor F on a category C is an initial object in the category of algebras of F .

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Category of algebras of F :

- ▶ Objects: alg , alg' , \dots

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Category of algebras of F :

- ▶ Objects: $\text{alg}, \text{alg}', \dots$
- ▶ Arrows : structure preserving maps (homomorphisms) from an algebra to another

Homomorphisms between two algebras.

An arrow in the category of F-algebras of a given endofunctor e.g. between $(\text{Natural}, \text{alg})$ and $(\text{String}, \text{alg}')$ is a function mapping the carrier in the underlying category $(\text{Hask}, \text{hom} : \text{Natural} \rightarrow \text{String})$, such that the following square commutes:

$$\begin{array}{ccc} F(\text{Natural}) & \xrightarrow{F(\text{hom})} & F(\text{String}) \\ \text{alg} \downarrow & & \downarrow \text{alg}' \\ \text{Natural} & \xrightarrow{\text{hom}} & \text{String} \end{array}$$

Figure 2:

That is to say that

```
fNat :: F Natural
fNat = Succ 1

hom :: Natural -> String
hom n = timesN n "QUACK" ++ "!"

> alg fNat -- 2
> fmap hom fNat -- Succ "QUACK!"
> hom $ alg fNat -- "QUACKQUACK!"
> alg' $ fmap hom fNat -- "QUACKQUACK!"
```

$$\text{hom} \circ \text{alg} \equiv \text{alg}' \circ \text{fmap hom}$$

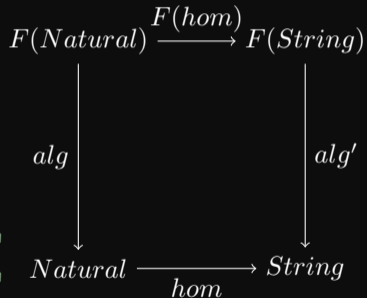


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An initial algebra for an endofunctor F on a category C is an initial object in the category of algebras of F .

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Category of algebras of F :

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Initial:

- ▶ There is a unique morphism from the initial algebra to *all other algebras*.

What fits?

- ▶ The carrier must not “lose” any information, or there is some algebra that it cannot map to.

$$X \text{ initial} \implies F(X) \cong X$$

$$\begin{array}{ccc} F \text{ Init}F & \xrightarrow{F(\text{hom})} & F(\text{String}) \\ \text{alg} \downarrow & & \downarrow \text{alg}' \\ \text{Init}F & \xrightarrow{\text{hom}} & \text{String} \end{array}$$

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What fits?

- ▶ The carrier must not “lose” any information, or there is some algebra that it cannot map to.
- ▶ The carrier can't add information, or the morphism won't be unique.

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- ▶ The algebra must have type: $F \text{ InitF} \rightarrow \text{InitF}$

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- ▶ Lambek's theorem says that if there is an initial object, it is isomorphic to the carrier via the algebra

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- ▶ Lambek's theorem says that if there is an initial object, it is isomorphic to the carrier via the algebra
- ▶ `data InitF = InitF (F InitF)`

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More generally...

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```
data Fix f = Roll { unRoll :: f (Fix f) }  
type InitF = Fix F  
Roll :: F InitF -> InitF  
unRoll :: InitF -> F InitF
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data Fix f = Roll { unRoll :: f (Fix f) }  
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```

```
fix2 :: InitF  
fix2 = Roll $ Succ $ Roll $ Succ $ Roll Zero
```

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data Fix f = Roll { unRoll :: f (Fix f) }  
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Roll :: F InitF -> InitF  
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```

```
fix2 :: InitF  
fix2 = Roll $ Succ $ Roll $ Succ $ Roll Zero
```

If this is the initial object in the category of algebras, there must be a unique arrow from `InitF` to every algebra:

$$\forall \text{algebras } \exists \text{hom} : \text{InitF} \rightarrow \text{carrier of algebra}$$

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{Init}F \rightarrow \text{carrier of algebra}$

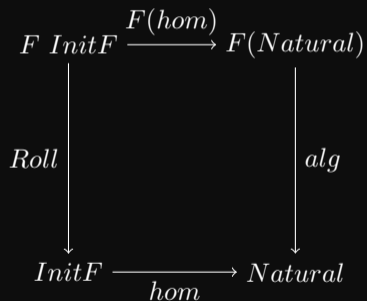
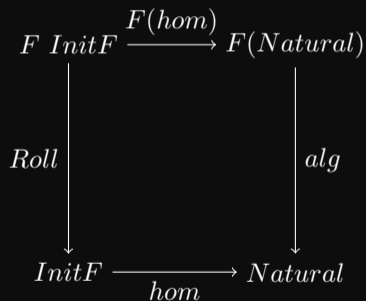


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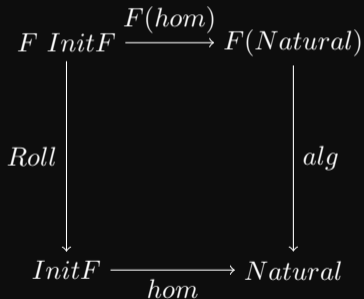


```
Roll :: F InitF -> InitF
```

Figure 5:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{InitF} \rightarrow \text{carrier of algebra}$



```
Roll :: F InitF -> InitF
```

```
hom :: InitF -> Natural
```

Figure 5:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{InitF} \rightarrow \text{carrier of algebra}$

```
unRoll :: InitF -> F InitF
```

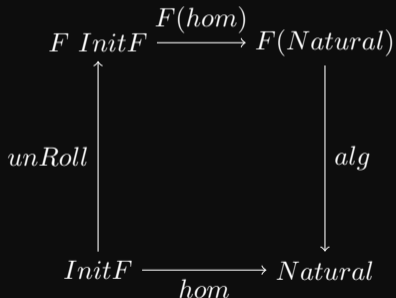
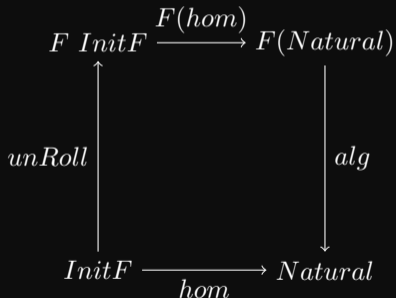


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```
unRoll :: InitF -> F InitF
```

```
hom :: InitF -> Natural
```

```
hom = alg . fmap hom . unRoll
```

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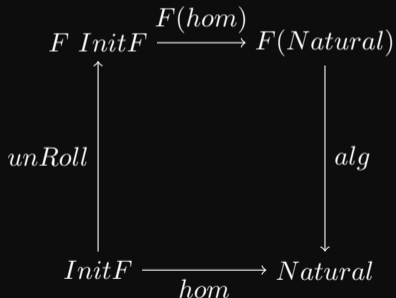


Figure 6:

```
unRoll :: InitF -> F InitF
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```
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```

```
cata :: Functor f
```

```
    => (f a -> a) -> Fix f -> a
```

```
cata alg =
```

```
    alg . fmap (cata alg) . unRoll
```

```
hom = cata alg
```


Evaluation of cata

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cata :: Functor f
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```
alg $ Succ $ alg $ Zero
```

This is recursion in a general sense

```
data Nat a = Succ a | Zero
```

This is recursion in a general sense

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data String a = Cons Char a | End
```


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data BinaryTree a = Branch a a | Tip
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```

```
data Group a = Action a a | Inv a | Unit
```

Hutton's razor - final tagless

```
class Calculator a where
  lit :: Int -> a
  add :: a -> a -> a
  mult :: a -> a -> a
```

```
instance Calculator Int where
  lit = id
  add = (+)
  mult = (*)
```

```
instance Calculator String where
  lit = show
  add s1 s2 = s1 ++ " + " ++ s2
  mult s1 s2 = s1 ++ " x " ++ s2
```

Hutton's razor - F algebra

```
data Calculator a = Lit Int | Add a a | Mult a a deriving Functor
```

```
evalAlg :: Calculator Int -> Int
```

```
evalAlg (Lit i) = i
```

```
evalAlg (Add i1 i2) = i1 + i2
```

```
evalAlg (Mult i1 i2) = i1 * i2
```

```
ppAlg :: Calculator String -> String
```

```
ppAlg (Lit i) = show i
```

```
ppAlg (Add s1 s2) = s1 ++ " + " ++ s2
```

```
ppAlg (Mult s1 s2) = s1 ++ " x " ++ s2
```

```
pp :: Fix Calculator -> String
```

```
pp = cata ppAlg
```

Damn the torpedos, flip the arrows

```
alg :: F Natural -> Natural
alg Zero = 0
alg (Succ n) = n + 1
```

Damn the torpedos, flip the arrows

```
alg :: F Natural -> Natural
alg Zero = 0
alg (Succ n) = n + 1
```

```
coalg :: Natural -> F Natural
coalg 0 = Zero
coalg n = Succ (n - 1)
```

Damn the torpedos, flip the arrows

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alg :: F Natural -> Natural
alg Zero = 0
alg (Succ n) = n + 1
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coalg :: Natural -> F Natural
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```

For a category C and endofunctor F a co-algebra of F is an object X in C and a morphism:

$$coalg : X \rightarrow F(X)$$

Morphisms on coalgebras: co all of the things!

$$\begin{array}{ccc} F(\text{Natural}) & \xrightarrow{F(\text{hom})} & F(\text{String}) \\ \downarrow \text{alg} & & \downarrow \text{alg}' \\ \text{Natural} & \xrightarrow{\text{hom}} & \text{String} \end{array}$$

Figure 7:

Morphisms on coalgebras: co all of the things!

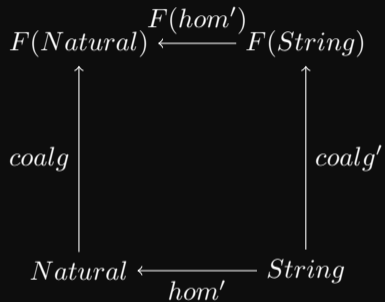
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$$\begin{array}{ccc} F(\text{Natural}) & \xleftarrow{F(\text{hom}')} & F(\text{String}) \\ \uparrow \text{coalg} & & \uparrow \text{coalg}' \\ \text{Natural} & \xleftarrow{\text{hom}'} & \text{String} \end{array}$$

Figure 8:

E.g.



```
coalg :: Natural -> F Natural
coalg 0 = Zero
coalg n = Succ (n - 1)
```

Figure 9:

E.g.

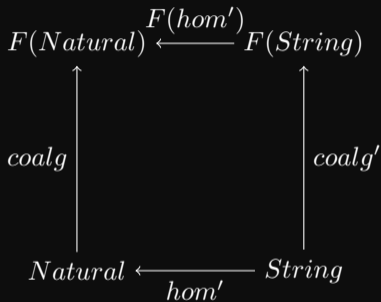
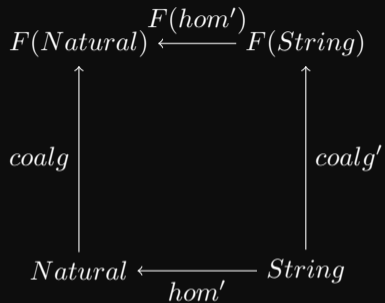


Figure 9:

```
coalg :: Natural -> F Natural
coalg 0 = Zero
coalg n = Succ (n - 1)
```

```
coalg' :: String -> F String
coalg' "!" =
  Zero
coalg' ('Q':'U':'A':'C':'K':xs) =
  Succ xs
```

E.g.



```
hom :: Natural -> String  
hom n = timesN n "QUACK" ++ "!"
```

Figure 10:

E.g.

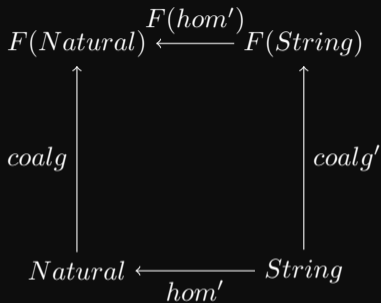
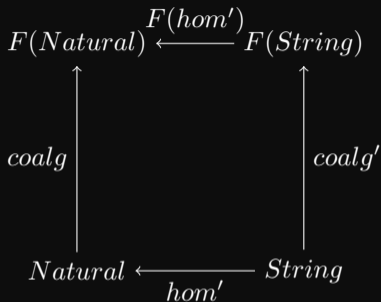


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```

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hom' :: String -> Natural
hom' str =
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    `div` 5
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Figure 10:

```
> (hom' "QUACKQUACK!", coalg' "QUACKQUACK!")
(2, Succ "QUACK!")
> (coalg $ hom' "QUACKQUACK!", fmap hom' $ coalg' "QUACKQUACK!")
(Succ 1, Succ 1)
```

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{carrier of algebra} \rightarrow \text{Term}F$

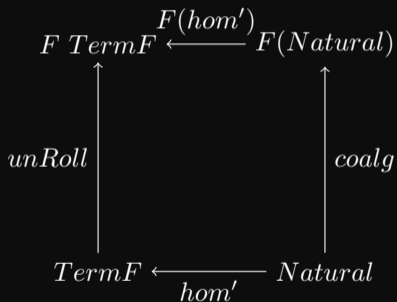
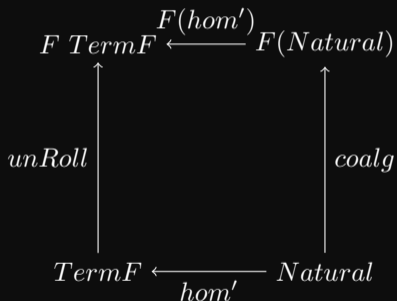


Figure 11:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{carrier of algebra} \rightarrow \text{TermF}$

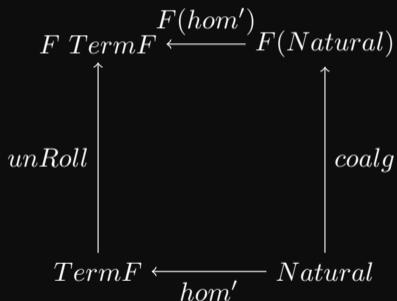


```
type TermF = InitF
unRoll :: TermF -> F TermF
```

Figure 11:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{carrier of algebra} \rightarrow \text{TermF}$



```
type TermF = InitF
unRoll :: TermF -> F TermF
```

```
hom' :: Natural -> TermF
```

Figure 11:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{carrier of algebra} \rightarrow \text{TermF}$

```
Roll :: F TermF -> TermF
```

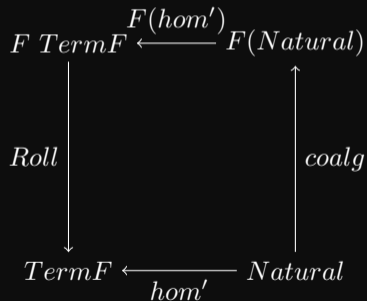
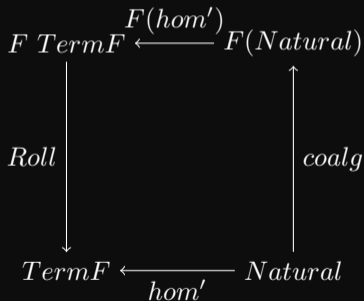


Figure 12:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{carrier of algebra} \rightarrow \text{TermF}$



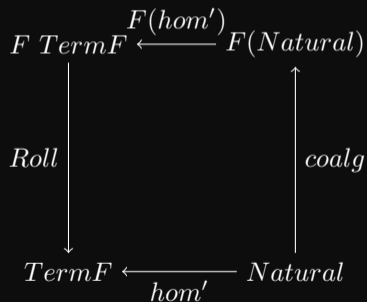
```
Roll :: F TermF -> TermF
```

```
hom' :: Natural -> TermF
```

Figure 12:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{carrier of algebra} \rightarrow \text{TermF}$



```
Roll :: F TermF -> TermF
```

```
hom' :: Natural -> TermF
```

```
hom' = Roll . fmap hom' . coalg
```

Figure 12:

The unique homomorphism

$\forall \text{algebras } \exists \text{hom} : \text{carrier of algebra} \rightarrow \text{TermF}$

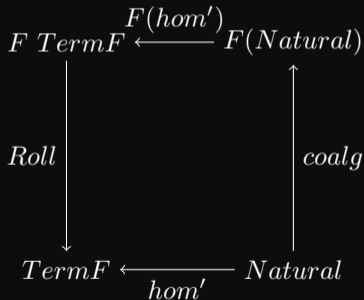


Figure 12:

```
Roll :: F TermF -> TermF
```

```
hom' :: Natural -> TermF
```

```
hom' = Roll . fmap hom' . coalg
```

```
ana :: Functor f  
    => (a -> f a) -> a -> Fix f  
ana coalg =  
    Roll . fmap (ana coalg) . f
```

Whilst we're here

```
cata :: Functor f => (f a -> a) -> InitF -> a  
cata alg = alg . fmap (cata alg) . unRoll
```

```
ana :: Functor f => (a -> f a) -> a -> TermF  
ana coalg = Roll . fmap (ana coalg) . coalg
```

Whilst we're here

```
cata :: Functor f => (f a -> a) -> InitF -> a
cata alg = alg . fmap (cata alg) . unRoll
```

```
ana :: Functor f => (a -> f a) -> a -> TermF
ana coalg = Roll . fmap (ana coalg) . coalg
```

```
cata alg' $ ana coalg' $ hom 3
> "QUACKQUACKQUACK!"
```


Whilst we're here

```
cata :: Functor f => (f a -> a) -> InitF -> a
cata alg = alg . fmap (cata alg) . unRoll
```

```
ana :: Functor f => (a -> f a) -> a -> TermF
ana coalg = Roll . fmap (ana coalg) . coalg
```

```
cata alg' $ ana coalg' $ hom 3
> "QUACKQUACKQUACK!"
```

```
hylo :: Functor f => (f b -> b) -> (a -> f a) -> a -> b
hylo alg coalg = alg . fmap (hylo alg coalg) . coalg
```

Whilst we're here

```
cata :: Functor f => (f a -> a) -> InitF -> a
cata alg = alg . fmap (cata alg) . unRoll
```

```
ana :: Functor f => (a -> f a) -> a -> TermF
ana coalg = Roll . fmap (ana coalg) . coalg
```

```
cata alg' $ ana coalg' $ hom 3
> "QUACKQUACKQUACK!"
```

```
hylo :: Functor f => (f b -> b) -> (a -> f a) -> a -> b
hylo alg coalg = alg . fmap (hylo alg coalg) . coalg
```

```
hylo alg' coalg' $ hom 3
> "QUACKQUACKQUACK!"
```